Vedic Mathematics: Expanding the View

In primary school mathematics, a sutra appears as an instruction. When multiplying 98 by 97, *All from nine and the last from ten* tells us to work with the deficiencies from 100. Working through a page of such sums, a child practices the same sutra in a rather special environment, and is introduced gently to the subtlety of mathematics. The deficiency 02 is after all nowhere visible in a row of 98 milk bottles, but even so it is the key to the calculation.

With secondary school or university mathematics we have a very different situation. We do not have a pre-assigned method for every possible problem. It may seem to the student, therefore, that Vedic Mathematics is lacking because there is nothing to say which sutra should be applied. The same difficulty arises with conventional textbooks, which give standard methods for standard problems, but cannot in the nature of things cover everything. How exactly do the sutras help us here?

Mathematics literature concentrates in the main on the finished result. It is a common complaint that the polished research paper leaves no clue as to how the new theorem was first conceived. But in many ways the genesis of the idea is the most interesting part. The formal proof is fine for the archives, but the sequence of logical steps, on its own, can miss the point, like a joke which has to be explained.

Fortunately, though, there are several excellent books which focus on the more subtle question of how the mind addresses a problem. These perhaps can help to broaden our appreciation of the sutras. One such is *How To Solve It: A New Aspect of Mathematical Method*, by Professor G Polya. Here he is describing one of the stratagems that may be brought to bear:

An insect tries to escape through the windowpane, tries the same over and over again, and does not try the next window which is open and through which it came into the room. A man is able, or at least should be able, to act more intelligently……going round an obstacle that cannot be overcome directly, devising a suitable auxiliary problem when the original problem appears insoluble. To devise an auxiliary problem is an important operation of the mind. To raise a clear cut new problem subservient to another problem, to conceive distinctly as an end what is a means to another end, is a refined achievement of the intelligence……

He gives the example of a quartic equation: find *x*, satisfying the equation

$$x^4 - 13x^2 + 36 = 0$$

If we observe that $x^4 = \left(x^2\right)^2$ we may see the advantage of introducing $y = x^2$.

We obtain now a new problem: find *y*, satisfying the equation

$$y^2 - 13y + 36 = 0.$$

The new (and much simpler) problem is an auxiliary problem, which we use to solve the main problem.

The crucial step is to "observe" the relevance of the substitution $y = x^2$ to our equation. How does this come about? The first step, says Polya, is to understand the problem; that is, explore it, be familiar with it, recognise its special features. Our original equation was categorised as a quartic, but far more relevant to its solution is the absence of terms in *x* and x^3.

Here is another of his examples. Suppose we have to prove the formula

$$V = \frac{\left(a^2 + ab + b^2\right)h}{3}$$

for the volume of the frustrum of pyramid with a square base. Its height is h, its lower base has side *a*, its upper base has side *b*. Before starting on serious calculation we may consider the shape of the formula. For example, each term consists of three lengths multiplied together, so that it is certainly a reasonable candidate for a volume. We can consider the special case, $b = a$. In this case $V = a^2 h$, which is the correct volume for the resulting rectangular box. Or we may set *b* =

0, so that $V = a^2h/3$, which is the correct formula for the volume of a complete pyramid. Experiment of this kind get us, so to say, on speaking terms with the formula. Indeed consideration of the complete pyramid will bring us very close to the realization that the desired volume V is in fact the difference in volume of two separate complete pyramids, one with base of side a and the other with base of side b. This is the key idea, following which we fill in the details at our leisure.

These examples help us with the question, what sutra should I use? The problem itself will tell us. The quartic equation cries out to be transformed into an easily soluble quadratic. The Vedic mathematician will appreciate this as *Transpose and apply*. And the truncated pyramid begs to be completed, even if only in mind. The sutra here is *By completion and non-completion*. To understand the sutras is to understand the language in which the problem speaks.

W.W.Sawyer writes about the significance of patterns in his *Prelude to Mathematics*, again from the mathematician's inside view. He offers us the simultaneous equations

$$6751x + 3249y = 26751$$
$$3249x + 6751y = 23249$$

What clues are there to suggest a line of attack? Most striking is the pattern

$$\begin{matrix} * & \# \\ \# & * \end{matrix}$$

which appears on the left-hand sides of the equations. Looking at this pattern, says Sawyer, he finds himself reasoning that the two equations are as good as each other. If we re-christen x as y, and y as x, the left hand sides are just interchanged. It would be unfair to do anything to one equation that we do not do to the other.

What operations are there which preserve an even handed approach? Adding the two equations is an obvious candidate. And subtraction also, he argues, is no sin against symmetry. The equation with the minus sign need not complain of injustice because equation (1) – equation (2) arrives at essentially the same statement as equation (2) – equation (1) . Following this lead, then, we find

$$10000x + 10000y = 50000$$
$$2502x - 2502y = 2502$$

or $x + y = 5$, $x - y = 1$, from which $x = 3$, $y = 2$. Again it is the shape and pattern of the problem which has shown us the route to the solution, which might be expressed in sutra form as *By addition and subtraction*.

All these are simple examples, chosen by our authors for easy demonstration. But the principle remains valid at all levels. On a slightly more exalted plane, consider a question set in a recent STEP (Sixth Term Examination Paper). Mathematical Olympiads aside, this is perhaps the most difficult exam that an English school student can encounter. The examiners, with their customary delight in principles and indifference to practicalities, suppose the existence of four mountain villages, situated at the corners of a regular tetrahedron. Each village is connected to each of the others by a road and each road has the same probability p of being blocked by snow. What is the probability that each village can still communicate, directly or indirectly, with all of the others?

It is important, even in an exam, to follow Polya's advice and take the trouble to explore the question. Rather than go for the full solution, all at once, we might ask how many roads must be cut off before communication fails. With 0, 1 or 2 roads blocked our villagers will always reach their destinations. With 6, 5, or 4 roads blocked there will always be at least one village inaccessible. Only in the case when 3 roads are blocked is more detailed analysis needed, and even this is not difficult. So it will pay to consider each of these cases in turn, and combine the individual calculations into the final result at the end. Here is the sutra, *By elimination and retention*.

Insights like these may take some time to arrive, and, once they have come, need many words to be explained, but their nature is of immediacy, illumination and certainty. In this regard Polya quotes Aristotle on "sagacity……..the perception of the essential connection in an inappreciable time". There are many examples of such illumination, from Archimedes' bath onwards, in scientific and mathematical history. Here is an instance recounted in Hadamard's *The Psychology of Invention in the Mathematical Field*. The mathematician Henri Poincare is addressing the Society of Psychology in Paris. His subject is the nature of mathematical discovery, rather than the technical aspects of what is discovered, so he first of all warns his audience. *"I shall say, for example, that I have found the demonstration of such a theorem under such circumstances. This will have a barbarous name, unfamiliar to many, but that is unimportant; what is of interest for the psychologist is not the theorem, but the circumstances."*

Poincare recounts his study of Fuchsian functions. At first he spends a fortnight trying to prove that there could be no such functions, an idea which would eventually turn out to be false. Indeed, during a sleepless night, he manages to find one example.

Just at this time, I left Caen, where I was living, to go on a geologic excursion under the auspices of the School of Mines. The incidents of my travel made me quite forget my mathematical work. Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it, that the transformations that I had used to define the Fuchsian functions were identical with those of non-Euclidean geometry. I did not verify the idea; I should not have had time, as, on taking my seat in the omnibus, I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience's sake, I verified the result at my leisure.

Then I turned my attention to the study of some arithmetical questions apparently without much success and without a suspicion of any connection with my preceding researches. Disgusted with my failure, I went to spend a few days at the seaside and thought of something else. One morning, walking on the bluff, the idea came to me, with the same characteristics of brevity, suddenness and immediate certainty, that the arithmetic transpositions of indefinite ternary forms were identical with those of non-Euclidean geometry.

Apparently from nowhere, then, the alternative route is presented: *Transpose and apply*.

Poincare is speaking of mathematics, and perception, at a very high level, but the appreciation of simplicity and economy in a mathematical solution is a natural human response. As Prof Sawyer remarks of his simultaneous equations, described above, "*children can find real pleasure in the experience of solving such a problem.*" The sutras focus attention on the creative side of mathematics and offer a way to refine and deepen our experience.

For one last example, take a problem set by Lewis Carroll under his real name of Charles Dodgson, lecturer in Mathematics at Christ Church, Oxford. One of a series of light-hearted puzzles, it concerns a group of Chelsea Pensioners (war veterans), of whom 70% have lost an eye, 75% an ear, 80% an arm, and 85% a leg; what percentage, at least, have lost all four?

Several students wrote in with their solutions and Dodgson publishes one of them as being better – that is, neater and more satisfying – than his own:

"*Adding the wounds together, we get 70 + 75 + 80 + 85 = 310, among 100 men, which gives three wounds to each, and four to ten men. Therefore the least percentage is ten*."

Although it is a simple problem there is something pleasingly direct about the argument from the totality of the wounds, irrespective of who actually received them. This suggests a sutra at work. Is it *Vyashti-Samashti*, which translates as *individuality-totality*?

There remains much work to be done in exploring the sutras and their applications, but the obvious direction lies in the mental processes which arise when a mathematical problem or question is being addressed.

J.Pickles, June 2001