Swami Tirtha's Crowning Gem

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Abstract

In his book Sri Bharati Krishna Tirthaji describes a division technique which he calls the "Crowning Gem" of Vedic Mathematics, the essential feature being that the first digit of the divisor is used to provide all subsequent digits when combined appropriately with the other digits involved. He then goes on to find square and cube roots in which the first digit of the answer is used in a similar way. The main Sutra in use for this is Ūrdhva Tiryagbhyām: Vertically and Crosswise. This approach can be developed considerably, so that powers and roots of numbers and polynomial expressions can be found, and polynomial equations can be solved to any accuracy, using the Vertically and Crosswise pattern. This paper describes this pattern, initially expressed as a formula. The operation of this pattern is then demonstrated in finding powers, and then used in reverse to find roots and solutions to polynomial equations.

Introduction

Sri Bharati Krishna Tirthaji describes the straight division method he gives as the "Crowning Gem" of Vedic Mathematics. He also shows how to find square roots in one line and extends this to cube roots, but still uses the first digit as a key to getting further digits.

This can be taken further: to powers and roots of numbers and polynomial expressions in general, and the general solution of polynomial equations. Here we illustrate the method by introducing a pattern which is used to cube a number, then applied in reverse to get the cube root back. Finally a cubic equation will be solved.

Duplex

Tirthaji introduces us to the Duplex, which helps in finding squares and square roots. They can also be used in the solution of quadratic equations.

In squaring say 134, we find successive Duplexes, and you may be familiar with these.
We take the square of the first digit: 1^2.
Then twice the product of the first two digits: $2\times1\times3$.
Then twice the product of the outer pair of the three digits plus the square of the middle digit: $2\times1\times4 + 3^2$.

Each of these successive Duplexes is one order of magnitude less than the previous one.

So in squaring *abcd* . . . , a^2 has the highest order,
$2ab$ the next highest,
$2ac + b^2$ the next highest
and so on.

Notice that the Duplexes can be generated from the first one. That is, we start with a^2 and for the next Duplex bring one of the *a*s down to *b*. So instead of *aa* we have *ab*, and the 2 is needed because we can have *ab* or *ba*.

Continuing, we can use *ab* to get the next lower Duplex by taking the *a* in *ab* down to *b* (to get b^2), or by taking the *b* down to *c* (to get *ac* or *ca*).

To extend beyond this: to cubing, cube roots and cubic equations and higher order equations, we need to extend the Duplex idea. So we get Triplexes, Quadruplexes etc.

Triplex

For cubing we use Triplexes.
So in cubing a number *abcd* . . . , a^3 will have the highest order.
The next Triplex can be generated from a^3 by taking one of the *a*s down to *b* to get $3a^2b$.

And this method of generating Triplexes can be continued for further Triplexes and for the higher order Quadruplexes and so on.

The Coefficients

There is also an easy way to get the coefficients. The coefficient for the term $p^m q^n$ is $\frac{(m+n)!}{m!n!}$, which for p^2q would be $\frac{(2+1)!}{2!1!}$ which is 3.

Just as practice is needed to feel comfortable with Duplexes, we need to work with the Triplexes to get the feel for them.

The Pattern/Formula

The pattern for obtaining powers and roots is given by:

$$
\begin{aligned}
f(x) = \quad & f(a) \; + \\
& f'(a)b \; + \\
& f'(a)c + \frac{f''(a)}{2!}D_1 \; + \\
& f'(a)d + \frac{f''(a)}{2!}D_2 + \frac{f'''(a)}{3!}T_1 \; + \\
& f'(a)e + \frac{f''(a)}{2!}D_3 + \frac{f'''(a)}{3!}T_2 + \frac{f^{iv}(a)}{4!}Q_1 \; + \\
& \; . \;\; . \;\; . \;\; .
\end{aligned}
$$

where $f(x) = x^p$, p is a positive integer > 1,
$x = a\times10^m + b\times10^{m-1} + c\times10^{m-2} + \ldots$ is a decimal number. So *abc* . . . are the successive digits of the number sought, or whose power is required.

D_n is the nth Duplex of all of the digits of x after the first,
T_n is nth Triplex of all of the digits of x after the first,
and so on for Quadruplexes Q_n, Pentaplexes P_n etc.

Each line on the right-hand side of the above equation represents one order of magnitude higher than the line below it.

f can also be a polynomial in the above equation but then we require that $1 < x < 10$ so that the various ‘Plexes’ can be combined at each step. Polynomials with roots outside this range can be easily transformed to conform with this requirement.

Note some features of the formula:
* Every term after the first on the right-hand side consists of a product of two numbers
* $f'(a)$ multiplies b, c, d etc. successively on each row after the first.
* In addition to $f'(a)$ we have $\frac{f''(a)}{2!}$, $\frac{f'''(a)}{3!}$ etc. We will call these our ‘multipliers’.
* Once we know the first digit, a, we know the values of $f'(a)$, $\frac{f''(a)}{2!}$, $\frac{f'''(a)}{3!}$ etc. So we can set up with those multipliers right at the outset, just as Tirthaji puts double the first digit on one side to act as a constant divisor when finding square roots.

* $\frac{f''(a)}{2!}$ always multiplies a Duplex, just as $\frac{f'''(a)}{3!}$ always multiplies a Triplex. Etc.

When finding **powers** we are simply evaluating the right-hand side of the equation.

When finding **roots and solving polynomial equations** we are trying to find, a, b, c, etc. and $f'(a)$ acts as a constant divisor because it multiplies b, c, d etc. in turn. So we are subtracting all the terms after the first in each row then dividing by $f'(a)$ to get b, c, d etc. in turn.

Cubing

Suppose we want **24^3**. We have $f(x) = x^3$, $a = 2$ and $b = 4$.

Then $\boldsymbol{f'(a)} = 3a^2 = 1\mathbf{2}$,
$\frac{f''(a)}{2!} = 3a = \mathbf{6}$ (this is the Duplex multiplier),
$\frac{f'''(a)}{3!} = \mathbf{1}$ (this is the Triplex multiplier).

The first entry on our answer row is the cube of the first digit: $2^3 = \mathbf{8}$:

2	4 .	0	0	. . . . given number
$f_n =$	12	6 **D**	1 **T**	. . . . multipliers
8				. . . . answer

The **D** and **T** above are to remind us that the ‘6’ is a Duplex multiplier and the ‘1’ is a Triplex multiplier.

Next, multiply vertically as before: 12×4 = **48**.

2	4 .	0	0
f_n =	12	6	1
8	48		

The crosswise step is shown below: 12 multiplies 0,
and 6 multiplies the first Duplex (the Duplex of 4, which is 16).
So we have 12×0 + 6×16 = **96**.

2	4 .	0	0
f_n =	12	6	1
8	48	96	

For the last step, 12 multiplies 0,
6 multiplies the second Duplex (the Duplex of 40 is 0),
and the 1 multiplies the Triplex of 4 (the Triplex of 4 is 64).
So we have 12×0 + 6×0 + 1×64 = **64**.

2	4 .	0	0
f_n =	12	6	1
8	48	96	64

Combining these four numbers from right to left (or left to right):

8			
4	8		
	9	6	
		6	4
13,	8	2	4

we get 13,824.

Therefore $\mathbf{24^3 = 13{,}824}$.

It may be possible to get the answer more easily in this case but this example is to illustrate the pattern for obtaining powers. In fact numbers with any number of digits can be raised to any positive integral power like this.

Also it is possible to work from right to left instead, which would have the advantage of smaller multipliers if the last digit is smaller than the first.

But best is to work from both ends as that avoids the bigger cross-products, some parts of which are zero.

Cube Root

Now let us reverse what we just did and get the cube root of 13,824.

We see immediately that the answer will have two digits before the decimal point, and that the first digit must be 2, since $2^3 = 8$ is the first cube below 13.

And there will be a remainder of 5, since 13 – 8 = 5.

Since the first digit is again 2, $a = 2$, and our multipliers will be the same as before: 12, 6 and 1.

But the 12 now acts as a divisor so it goes at the left.

	13	$_{5}$**8**	2	4	. . . . given number
12)		6	1		. . . . multipliers
					Initial chart
	2				. . . . answer

Then, dividing 12 into 58 we get **4** remainder **10**:

	13	$_{5}$8	$_{10}$**2**	4	
12)		6	1		Second digit
	2	**4**			

We have our 24 back as expected. But we can continue the calculation: to show the pattern, and to show the remaining parts of 13,824 are exhausted by this process.

So next we have 102 – 6×D(4) = 6,
and 6 ÷ 12 = **0** remainder **6**:

$$\begin{array}{r|llll} & & & & \\ 13 & {}_{5}8 & {}_{10}2 & {}_{\mathbf{6}}\mathbf{4} & \\ 12) & \;6 & \;1 & & \\ \hline 2 & 4\;. & \mathbf{0} & & \end{array}$$

Third digit

Finally, 6×D(40) = 0,
1×T(4) = 64
and 64 – 64 = 0, which fully exhausts 13824.

$$\begin{array}{r|lllll} 13 & {}_{5}8 & {}_{10}2 & {}_{\mathbf{6}}4 & {}_{\mathbf{0}}\mathbf{0} \\ 12) & \;6 & \;1 & & \\ \hline 2 & 4\;. & 0 & \mathbf{0} & \end{array}$$

Fourth digit

So **the cube root of 13824 is 24**.

Following this pattern 4th, 5th etc. roots can be found whether perfect powers or not.

Cubic Equation

This follows the same lines as for the cube root: we find the first figure, then the multipliers, then apply the pattern.

Find a solution to $\boldsymbol{x^3 - 4x^2 + 12x = 32}$.

We find $x = 3$ gives the smallest remainder, so $a = \mathbf{3}$ and the remainder is
$32 - [3^3 - 4(3)^2 + 12(3)] = \mathbf{5}$.

$f(x) = x^3 - 4x^2 + 12x$

$f'(x) = 3x^2 - 8x + 12$. So $f'(3) = \mathbf{15}$.

$\frac{f''(a)}{2!} = 3a - 4 = \mathbf{5}$.

$\frac{f'''(a)}{3!} = \mathbf{1}$.

So 15 is the divisor, 5 is the Duplex multiplier and 1 is the Triplex multiplier.

We begin:

	32	. $_{5}$**0**	0	0	0
15)		**5**	**1**	0	
	3	.			

Initial chart

Next, 50 ÷ 15 = **3** remainder **5**:

	32	. $_{5}$0	$_{5}$**0**	0	0
15)		5	1	0	
	3	. **3**			

Second digit

Next, 50 – 5 × D(3) = 5.
5 ÷ 15 = **0** remainder **5**:

	32	. $_{5}$0	$_{5}$0	$_{5}$**0**	0
15)		5	1	0	
	3	. 3	**0**		

Third digit

Next, 5 × D(30) + 1 × T(3) = 27.
50 – 27 = 23.
23 ÷ 15 = **1** remainder **8**:

	32	. $_{5}$0	$_{5}$0	$_{5}$0	$_{8}$**0**
15)		5	1	0	
	3	. 3	0	**1**	

Fourth digit

Notice that we have got 4 figures and only had to go as far as the first Triplex.

Next, 5 × D(301) + 1 × T(30) = 30.
80 – 30 = 50.
50 ÷ 15 = **3** remainder **5**:

	32	. $_{5}$0	$_{5}$0	$_{5}$0	$_{8}$0	$_{5}$0
15)		5	1	0		
	3	. 3	0	1	**3**	

And so on.

We can similarly solve quartic, quintic etc. equations. Once the pattern is assimilated the procedure is straightforward.

There is further scope for research here. For example other functions, such as exponential and trigonometric functions can be expressed in polynomial form and so may be approachable in this way.

Summary

The process outlined here is the natural extension of Swami Tirtha's Crowning Gem of Vedic Mathematics. Its application is wide and it exhibits the usual 'Vedic' features of simplicity, directness and reversibility.

References

[1] Bharati Krsna Tirthaji Maharaja, " Vedic Mathematics", Motilal Banarasidas Publisher, Delhi, 1994.
[2] Williams, K. R., "The Crowning Gem", Inspiration Books, U.K., 2013