Easy Calculus

J.Pickles, July 2003

In [the horn, or the snail shell] the presently existing structure is, so to speak, partly old and partly new. It has been conformed by successive and continuous increments; and each successive stage of growth, starting from the origin, remains as an integral and unchanging portion of the growing structure.

[D'Arcy Thompson, On Growth and Form]

Growth and Expansion

In Vedic Mathematics Tirthaji describes the differential calculus as belonging under the sutra *Calanakalanabhyam*, 'By being moved and by setting in motion'. Let us consider growth and expansion as examples of movement.

Example 1

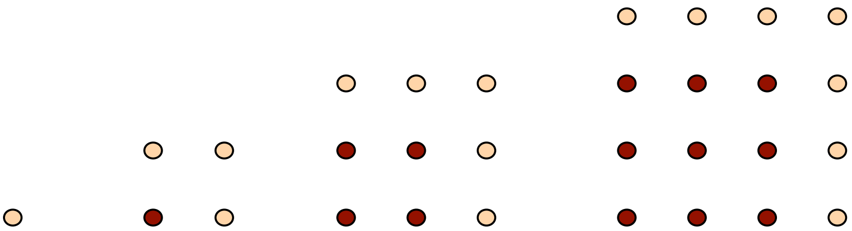
The Nautilus lives in the outermost chamber of its spiral shell. When it needs more room, it builds a new chamber and seals off the old one. The shell therefore grows by a process of successive addition.

Example 2

The sequence of natural numbers 1, 2, 3, …..unfolds according to the sutra *By one more than the one before*. One is added at every step.

Example 3

The sequence of square numbers provides another illustration.



The gnomon

The portion added at each stage, like the chamber of the nautilus, is called the gnomon (Greek for a carpenter's rule, like an L-shaped set square). In Example 2 the single unit is the gnomon while in Example 3 it is the L-shaped array illustrated above.

It is clear in each illustration that the whole figure is generated by successive addition of the gnomon. This is reflected in arithmetic statements like

1 = 1

2 = 1 + 1

3 = 1 + 1 + 1

4 = 1 + 1 + 1 + 1

for Example 2, and by the relationship between square numbers and sums of odd numbers demonstrated by Example 3.

1 = 1

4 = 1 + 3

9 = 1 + 3 + 5

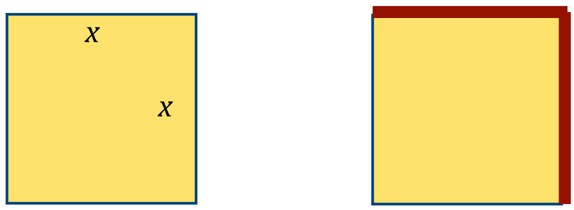
16 = 1 + 3 + 5 + 7 and so on.

Differential Calculus

Calculus describes the process of growth when it proceeds by continuous increments rather than by a series of distinct steps. The growth of a leaf might be an example. It grows in area by means of an expansion which shows itself at the perimeter.

Example 4

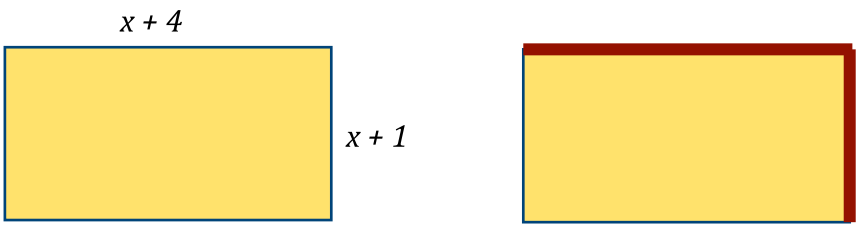
Consider again the growth of a square, but now suppose that the side of the square, instead of being measured in whole number units, as in Example 2, is able to grow continuously.



The Vedic sutra *Gunaka Samuccaya*, or *Sum of the factors*, defines the differential of the area of the square, x^2, as the sum of the factors, that is $x + x = 2x$.

The standard formulation of the calculus requires an abstract analysis of the relationship of the infinitesimal increase of the area of the square, x^2, which is engendered by an infinitesimal increase of the side x. But the sum of the factors rule refers the student directly to what is plainly visible in the diagram as the shaded part of the square's perimeter.

The same rule applies when the factors are unequal.



Here (by V*ertically and Crosswise*) the area is $(x+1)(x+4) = x^2 + 5x + 4$, and the differential is $(x+1) + (x+4) = 2x + 5$.

Example 5

The word *samuccaya*, translated above as *sum*, means literally *collecting up all together* and can embrace a wider scope of meaning than the simple sense of sum or total. As an example consider the growth of a cube of side x. Its volume, x^3, is the product of three factors: x times x times x. Here the differential is given by summing the product of each pair of factors: $(x \times x) + (x \times x) + (x \times x) = 3x^2$. We see that $3x^2$ is the area of the three 'growing' faces of the cube.

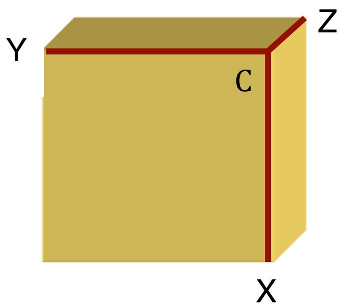
Again the rule applies even when the factors are unequal. For example, the differential of $(x+1)(x+2)(x+3)$ is

$(x+2)(x+3) + (x+1)(x+3) + (x+2)(x+3)$

which is still the area of the three growing faces.

Repeated Differentials

Looking again at the cube, we can see that each of the faces x^2, which manifest the growth of the cube, is itself growing, and has its own differential, 2*x*.



We say that the first differential of x^3 is $3x^2$, and that the differential of this differential, which is

$$2x + 2x + 2x = 6x$$

is the <u>second</u> differential of the original x^3.

In the diagram, the second differential can be identified with the edges *CX*, *CY*, *CZ* of the cube. The second differential comes out as 6*x*, rather than 3*x*, because each of these three edges belongs to two of the growing faces and each is therefore counted twice.

The <u>third</u> differential of x^3 is the differential of the second differential. The differential of a single growing edge, *x*, is simply 1, representing the dimensionless point at the end of the line. So the differential of 6*x* is 6, which represents the corner of the cube *C*, counted twice for each of three edges, that is six times altogether.

General Rule

Gunaka Samuccaya applied to the general n^{th} power x^n, which is the product of *n* factors *x*, requires that the factors are taken together in groups of (*n* –1). There are *n* such groups, since each of the n factors *x* is left out in turn. So the differential of x^n is nx^{n-1}. For example, the differential of x^5 is $5x^4$, the differential of $x = x^1$ is $x^0 = 1$. The differential of 1, or of any constant, is zero.

As a check, we can work out the differentials in Examples 4 and 5 in two different ways. Considering the product (x+1)(x+4), the differential can be calculated as either the sum of the factors

(x +1) + (x+4) = 2x + 5

or by taking the differential, term by term, of the product (x+1)(x+4) = x^2 + 5x + 4. The differential of the first term, x^2, is 2x. The differential of the second term, 5x, is 5 times the differential of x, or $5 \times 1 = 5$. And the differential of 4, which necessarily never changes, is zero. The total differential is confirmed as 2*x* + 5.

Likewise the differential of (x+1)(x+2)(x+3) = x^3 + $6x^2$ +11x + 6 can be calculated as

either (x+2)(x+3) + (x+1)(x+3) + (x+1)(x+2) = (x^2+5x+6) + (x^2+4x+3) + (x^2+3x+2) = $3x^2$+12x+11,

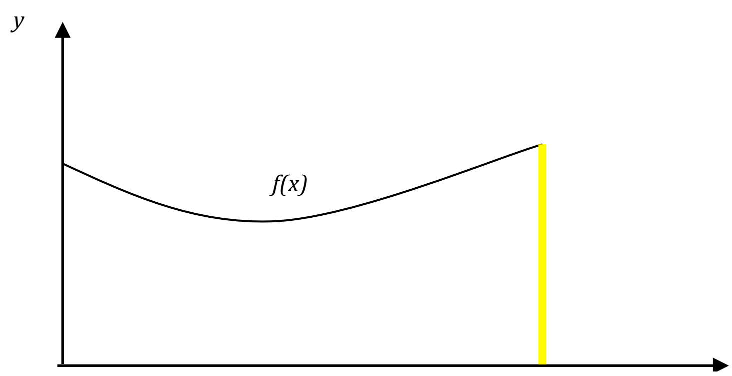
or as $3x^2$ +6×2x + 11×1 = $3x^2$+12x+11.

Integral calculus

The process of differentiation, as explained so far, identifies the growing margin of the complete form. Integration is the reverse process, generating the whole form from the growing margin. So for example the differential of x^3 is $3x^2$, and the integral of $3x^2$ is x^3. Similarly, the integral of 1 with respect to x is x, the integral of $8x^3$ with respect to x is $2x^4$, the integral of $x + 8x^3$ with respect to x is $\frac{1}{2}x^2 + 2x^4$.

Usually, the differential calculus is introduced as the process which finds the slope of a graph, while integration gives the area beneath it. This leaves open the question as to why these two processes should be the reverse of one another.

The answer, which is called the fundamental theorem of the calculus, is usually considered to be beyond the scope of the A level P1 student. Yet the explanation is very easy to see



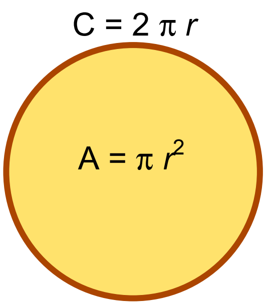
The area under the graph is the integral of f(x). As the domain of integration is extended, this area grows at its right hand edge, as indicated by the shaded line, and this growing edge represents the differential of the area. But the height of this line is of course f(x). So the differential of the integral of f(x) takes us back to f(x)!

Applications of Calculus

The simplicity of the Vedic approach to calculus allows its use in several elementary areas of mathematics.

Formulae for circle and sphere

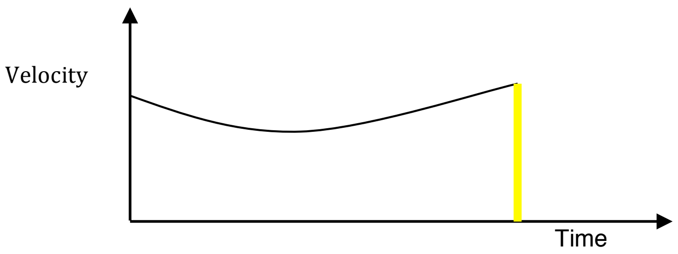
The well-known formulae for the circumference *C* and area *A* of a circle of radius *r* are $C = 2\pi r$, $A = \pi r^2$. The differential of *A* is just π times the differential of r^2, or π times 2r, or 2 π*r*, which is *C*. This is evident because the circumference is the boundary at which the area grows. Conversely the area *A* is the integral of *C*. We can visualize the cross-sectional area of a circular tree trunk as simply the totality of the tree rings.



In the same way the surface area S of a sphere of radius *r*, $S = 4\pi r^2$, is the differential of the volume $V = \frac{4}{3}\pi r^3$, and *V* is the integral of *S*.

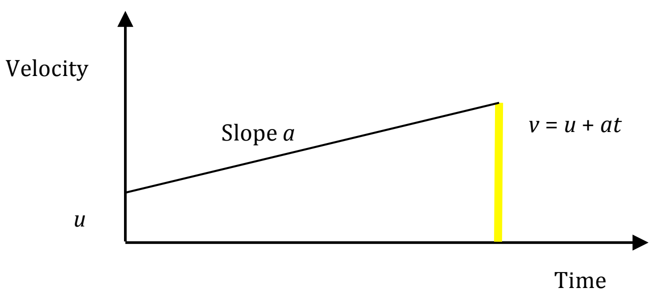
Velocity-time graphs

A natural application of calculus is to the study of motion and the relationship between position and velocity. If we consider motion along an axis x, velocity v can be regarded as the rate of change, or differential, of position, and we write v = dx/dt. Conversely changes in position are generated by movement so that position x may be regarded as the integral of velocity v. These relationships are illustrated by the velocity-time graph



Velocity is plotted on the vertical axis and the area under the graph then represents change in position, or displacement.

In the case of constant acceleration the velocity graph is a straight line as specified by the constant acceleration formula $v = u + at$.



The area under the graph, representing displacement, is the integral of velocity with respect to time. Hence the displacement is

$$s = \int (u + at)dt = ut + \frac{1}{2}at^2$$

which is the constant acceleration formula for s.

Approximations

The use of the differential calculus in the approximate solution of equations is illustrated by a story about Richard Feynman, the Nobel prize-winning physicist. Once, while at a restaurant, Feynman was challenged to an arithmetic contest by an abacus expert. The expert would use his abacus while Feynman would calculate mentally. The abacus man proposed that they should both calculate the cube root of 1729, the calculation of cube roots being the supreme accomplishment of the abacus world. As it happens, 1729 is very close to 1728, which Feynman knew was the cube of 12, since there are 1728 cubic inches in a cubic foot. He just needed to "grow" the 12 sufficiently for its cube to became 1729 instead of 1728. Picturing a cube of side $x = 12$, its differential $3x^2 = 3 \times 144 = 432$ represents the area of three growing faces. Approximately, an extra thickness of 1/432 is needed on each of these faces to create the additional 1 unit of volume required. So the increment in x is 1/432 = 0.0023… and the required cube root is 12.0023…

Readers may care to experiment with similar examples, such as say the calculation of $\sqrt{17}$, which must evidently be just a little bit greater than 4. The general principle is that if we require the solution of f(x) = a, and we know x_0 such that $f(x_0) = a - \delta$, where δ is small, then an approximate solution for x is $x_1 = x_0 + \delta/f'(x_0)$, where $f'(x_0)$ represents the differential of $f(x)$ at $x = x_0$.

The Newton-Raphson method repeats the approximation process, first finding x_1 from x_0, then getting a better approximation x_2 from x_1, and so on, to get very accurate results. Newton demonstrated how rapidly the accuracy increases by taking the cubic equation $x^3 - 2x = 5$ and starting with the initial value $x_0 = 2$. Then $f(x_0) = 4$ and $f'(x_0) = 3x_0^2 - 2 = 10$, so that $x_1 = 2.1$. By the third step of the calculation we have $x_3 = 2.09455148$, which is correct to 8 places of decimals. Try it!

Further Applications

At a more advanced level, the simple Vedic approach to calculus helps with topics like the Binomial Theorem, Maclaurin's Theorem, or Leibniz's theorem. These we leave to a later occasion…..